EFFECT OF COATING LAYER FOR SPHERES UNDER HERTZIAN CONTACT

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A model is constructed to analyze the effects of layer hardness and thickness upon contact stresses for the coated elastic sphere under normal loading. It is assumed that the layer is perfectly bonded to the elastic substrate and the radius of contact is very small compared to the radius of indenter. By following a linear theory of elasticity, Fredholm integral equation is developed and it is solved numerically. The resulting contact stresses are calculated at the layer surface as well as the layer-substrate interface. Also, the second invariant of the deviatoric stress tensor, $\sqrt{J_2}$ are calculated for various layer substrate combinations and for several layer thickness.

Key Words: Layer, Substrate, Coating, Fourier Integral Transform, Fredholm Integral Equation, Interfacial Stress, Deviatoric Stress

NOMENCLATURE

a : Radius of contact
R : Radius of sphere
H : Layer thickness
J_2 : The second invariant of the deviatoric stress tensor
p_z : Normal force
δ_{ap} : Relative approach of two spheres
Γ : Ratio of the shear modulus (μ_1/μ_0)
u_{rj} $u_{\theta j}$ u_{zj} : Displacement components
τ_{rrj} $\tau_{\theta\theta j}$ τ_{zzj} $\tau_{z\theta j}$ $\tau_{z\theta j}$: Stress components
μ_j : Shear modulus
ν_j : Poisson's ratio
p_j : Hydro-static pressure
($j=0$; The substrate
$j=1$; The layer)
$J_o(\xi r)$, $J_{-1/2}(\xi r)$: Bessel function
$sh(\xi r)=\mathrm{Sinh}(\xi r)$
$ch(\xi r)=\mathrm{Cosh}(\xi r)$

1. INTRODUCTION

For many tribological applications the production of better wear resistant materials requires hardening of surfaces. However, when a hardened surface do not have the desired properties, coated materials may be preferable. Some relevant examples of coated products are cemented carbide cutting tips (TiN, TiC), ball bearings, gears, high speed drills, milling cutters and many machine elements.

The state of stress that arises when two deformable bodies are pressed together by forces normal to the common tangent plane at the point of initial contact is of great technological interest and has a long history. In 1882, Hertz(Hertz, 1882) analyzed the problem of normal frictionless contact for isotropic, smooth homogeneous materials. Later, Mindlin (Mindlin, 1949) treated same case for rough bodies undergoing normal and tangential loading. Lysmer and Duncan (Lysmer and Duncan, 1972) published an extensive survey of the literature most of which deals with a uniform normal traction distributed over a circular surface region. In 1975, Goodman and Keer(Goodman and Keer, 1975) studied the case in which the deformation takes place in identical elastic surface layers of arbitary thickness bonded to a rigid sphere.

In this present analysis, their reasoning is extended to the case of an elastic layer with perfect adhesion to an elastically dissimilar substrate. By following a linear theory of elasticity, Fredholm integral equation is developed and it is solved numerically. Generally, material yielding is governed by the Von Mises yield criterion. In this analysis, the second invariant of the deviatoric stress tensor, $\sqrt{J_2}$, are calculated for various layer substrate combinations for the layer thickness $0.25<H/a<5$.

2. BASIC EQUATIONS AND DERIVATIONS

Two elastic spheres with elastically dissimilar surface layers are subjected to a normal force, p_z, and produces a resultant contact radius, a. It is assumed that the layers are perfectly bonded to the substrate. The geometry and the coordinate system is shown in Fig. 1 where the subscripts 1 and 0 represent the layer and substrate, respectively. The state of stresses and displacements in each of the layers and the substrates due to the normal loading are governed by a following linear theory of elasticity (Sneddon, 1951) as:

$$2\mu_a u_{ra}=\frac{\partial F_a}{\partial r}+z\frac{\partial G_a}{\partial r} \tag{1a}$$

$$2\mu_a u_{\theta a}=\frac{1}{r}\frac{\partial F_a}{\partial \theta}+\frac{z}{r}\frac{\partial G_a}{\partial \theta} \tag{1b}$$

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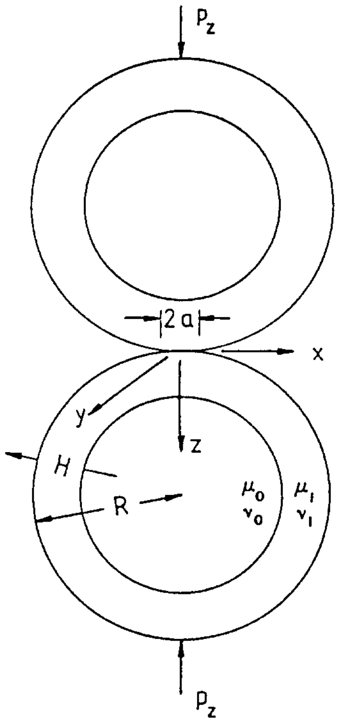


Fig. 1 Geometry of the problem

$$2\mu_a u_{za} = \frac{\partial F_a}{\partial z} - (3-4\nu_a)\,G_a + z\frac{\partial G_a}{\partial z} \tag{1c}$$

$$\tau_{zra} = \frac{\partial^2 F_a}{\partial r \partial z} - (1-2\nu_a)\frac{\partial G_a}{\partial r} + z\frac{\partial^2 G_a}{\partial z \partial r} \tag{1d}$$

$$\tau_{z\theta a} = \frac{1}{r}\frac{\partial^2 F_a}{\partial \theta \partial z} - (1-2\nu_a)\frac{1}{r}\frac{\partial G_a}{\partial \theta} + \frac{z}{r}\frac{\partial^2 G_a}{\partial \theta \partial z} \tag{1e}$$

$$\tau_{rra} = \frac{\partial^2 F_a}{\partial r^2} - 2\nu_a\frac{\partial G_a}{\partial z} + z\frac{\partial^2 G_a}{\partial r^2} \tag{1f}$$

$$\tau_{\theta\theta a} = \frac{1}{r^2}\frac{\partial^2 F_a}{\partial \theta^2} - 2\nu_a\frac{\partial G_a}{\partial z} + \frac{z}{r^2}\frac{\partial^2 G_a}{\partial \theta^2} \tag{1g}$$

$$\tau_{zza} = \frac{\partial^2 F_a}{\partial z^2} - 2(1-\nu_a)\frac{\partial G_a}{\partial z} + z\frac{\partial^2 G_a}{\partial z^2} \tag{1h}$$

$$(a=0,\ 1)$$

The harmonic functions $F_a(r,\theta,z)$ and $G_a(r,\theta,z)$ can be taken as

$$F_1(r,z) = \int_0^\infty [A(\xi)\,\mathrm{sh}(\xi z) + B(\xi)\,\mathrm{ch}(\xi z)]\,\xi^{-1} J_0(\xi r)\,d\xi \tag{2a}$$

$$G_1(r,z) = \int_0^\infty [C(\xi)\,\mathrm{ch}(\xi z) + D(\xi)\,\mathrm{sh}(\xi z)]\,J_0(\xi r)\,d\xi \tag{2b}$$

$$F_0(r,z) = \int_0^\infty U(\xi)\,[\mathrm{sh}(\xi z) - \mathrm{ch}(\xi z)]\,\xi^{-1} J_0(\xi r)\,d\xi \tag{2c}$$

$$G_0(r,z) = \int_0^\infty V(\xi)\,[\mathrm{sh}(\xi z) - \mathrm{ch}(\xi z)]\,J_0(\xi r)\,d\xi \tag{2d}$$

The boundary conditions for the problem are as follows:

$$\tau_{zr1}=0 \qquad z=0 \quad 0<r<\infty \tag{3a}$$

$$u_{z1}=w(r) \qquad z=0 \quad 0<r<a \tag{3b}$$

$$\tau_{zz1}=0 \qquad z=0 \quad a<r<\infty \tag{3c}$$

$$u_{r1}=u_{r0} \quad u_{z1}=u_{z0} \qquad z=H \quad 0<r<\infty \tag{4ab}$$

$$\tau_{zz1}=\tau_{zz0} \quad \tau_{zr1}=\tau_{zr9} \qquad z=H \quad 0<r<\infty \tag{4cd}$$

Once an arbitary coefficient $\varepsilon(\xi)$ is introduced such that

$$\tau_{zz1}(r,0) = \int_0^\infty \varepsilon(\xi) J_0(\xi r)\,d\xi \tag{5}$$

coefficients, $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$, $U(\xi)$ and $V(\xi)$ in Eq. (2) can be calculated in terms of $\varepsilon(\xi)$ as following:

$$[\mathrm{CDUV}]^T = \varepsilon(\xi)\{M\}^{-1}[\mathrm{sh}\beta\ \mathrm{ch}\beta\ \mathrm{sh}\beta\ \mathrm{ch}\beta]^T \tag{6abcd}$$

and

$$A=(1-2\nu_1)\,C,\quad B=2(1-\nu_1)\,D-\varepsilon(\xi) \tag{6ef}$$

where,

$$M=\begin{bmatrix} \beta\mathrm{sh}\beta-2(1-\nu_1)\mathrm{ch}\beta & \beta\mathrm{ch}\beta-(1-2\nu_1)\mathrm{sh}\beta & -\Gamma e^{-\beta} & -\Gamma\{\beta+(3-4\nu_0)\}e^{-\beta} \\ \beta\mathrm{sh}\beta-2(1-\nu_1)\mathrm{sh}\beta & \beta\mathrm{sh}\beta+2(1-\nu_1)\mathrm{ch}\beta & -\Gamma e^{-\beta} & \Gamma\beta e^{-\beta} \\ \beta\mathrm{sh}\beta & \beta\mathrm{ch}\beta-\mathrm{sh}\beta & -e^{-\beta} & -\{\beta+(1-2\nu_0)\}e^{-\beta} \\ \beta\mathrm{sh}\beta-\mathrm{sh}\beta & \beta\mathrm{ch}\beta & e^{-\beta} & \{\beta+2(1-\nu_0)\}e^{-\beta} \end{bmatrix}$$

Here, $\beta=\xi H$ and $\Gamma=\mu_1/\mu_0$.
In order to satisfy Eq. (3b) and Eq. (3c), the following dual integral equations are developed;

$$\int_0^\infty \varepsilon(\xi) J_0(\xi r)\,d\xi - \int_0^\infty \varepsilon(\xi) L(\beta,\nu_1,\nu_0) J_0(\xi r)\,d\xi = \frac{\mu_1}{1-\nu_1}w(r) \qquad 0<r<a \tag{7a}$$

$$\int_0^\infty \varepsilon(\xi) J_0(\xi r)\,\xi d\xi = 0 \qquad a<r<\infty \tag{7b}$$

with

$$L(\beta,\nu_1,\nu_0) = 1 + C(\xi) \tag{7c}$$

To solve Eq. (7a) and Eq. (7b), a technique due to Copson (Copson, 1961) is followed. When $\varepsilon(\xi)$ is expressed as

$$\varepsilon(\xi) = \xi^{1/2}\int_0^a t^{1/2}\phi_0(t) J_{-1/2}(\xi t)\,dt \tag{8}$$

then,

$$\tau_{zz1}(r,0) = \left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{1}{(a^2-r^2)^{1/2}}\phi_0(a) + \int_r^a \frac{[\phi_0(t)]'}{(t^2-r^2)^{1/2}}dt\right\} \qquad 0<r<a \tag{9a}$$

$$=0, \qquad a<r<\infty \tag{9b}$$

Now, Eq. (3c) is satisfied automatically, and for the pressure vanishing at the edge of the contact region r=a,

$$\phi_0(a) = 0 \tag{10}$$

Following Goodman and Keer (Goodman and Keer, 1975), the governing equation for ϕ_0 can be derived from Eq. (7a) as

$$\phi_0(s) - s^{1/2}\int_0^a t^{1/2}\phi_0(t)\,dt\int_0^\infty L(\beta,\nu_1,\nu_0)\,\xi J_{-1/2}(\xi s) J_{-1/2}(\xi t)\,d\xi = \left(\frac{2}{\pi}\right)^{1/2}\frac{\mu_1}{1-\nu_1}\frac{d}{ds}\int_0^s \frac{w(r)\,r}{(s^2-r^2)^{1/2}}dr \tag{11}$$

The resultant normal force, P_z, is

$$P_z=\int_0^{2\pi}\int_0^{a}-\tau_{zz1}(r,\theta,0)\,r\,dr\,d\theta$$
$$=4\left(\frac{\pi}{2}\right)^{1/2}\int_0^{a}\phi_0(t)\,dt \qquad (12)$$

It is assumed that the contact radius, a, is small compared with the radius of the spheres, R. According to Hertz theory, when the bodies are compressed by normal forces, the normal displacement at the boundary of the contact becomes

$$u_{z1}(r,\theta,0)=\frac{1}{2}[\delta_{ap}-r^2/R] \qquad 0<r<a \qquad (13)$$

where, δ_{ap} is the relative approach of two spheres. Using the following nondimensional definitions

$$s=\sigma a,\ t=\tau_a,\ \beta=\xi H,\ \delta_{ap}=C_1\frac{a^2}{R}$$
$$\phi_0(s)=\frac{a^2}{R}\frac{\mu_1}{(1-\nu_1)}\Phi_0(\sigma) \qquad (14)$$

Eq. (11) reduces to

$$\Phi_0(\sigma)-\frac{2}{\pi}\left(\frac{a}{H}\right)\int_0^{1}\Phi_0(\tau)\,d\tau\int_0^{\infty}L(\beta,\nu_1,\nu_2)$$
$$\cos\left(\beta\sigma\frac{a}{H}\right)\cos\left(\beta\tau\frac{a}{H}\right)d\beta$$
$$=\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{2(1-\nu_1)}(C_1-\sigma^2) \qquad (15)$$

The stress function Φ_0 and the relative approach C_1 can be determined simultaneously by means of solution of symmetric Fredholm integral equation in Eq. (15) and an auxiliary condition, Eq. (10). From Eq. (10) and Eq. (15), the contact radius, a, is determined: $a/R=C_2\left[\frac{1-\nu_1}{\mu_1}P_z/\pi a^2\right]$. From Eq. (9a), the maximum normal pressure in the contact region can be calculated: $\tau_{zz1(max)}=-C_3[P_z/\pi a^2]$. For a limiting case of an infinitely thick layer ($H/a=\infty$), the Hertz result is obtained : $C_1=1.0$, $C_2=3/8\pi$, and $C_3=1.5$. The numerical values of C_1, C_2 and C_3 are given in Fig 2, Fig. 3, and Fig 4 for various layer substrate combinations for the layer thickness ranging $0.2<H/a<5$. The stress components in the layer and the substrate are evaluated as follows:

$$\tau_{rri}(\rho,z/H)=P_z/\pi a^2\int_0^{\infty}\Omega_i(\rho,z/H)J_0\left(\beta\frac{a}{H}\rho\right)d\beta \qquad (16a)$$
$$\tau_{rzi}(\rho,z/H)=P_z/\pi a^2\int_0^{\infty}\Psi_i(\rho,z/H)J_1\left(\beta\frac{a}{H}\rho\right)d\beta \qquad (16b)$$
$$\tau_{\theta\theta i}(\rho,z/H)=P_z/\pi a^2\int_0^{\infty}\Big[\eta_i(\rho,z/H)\left(\frac{H}{a\rho}\right)J_1$$
$$\left(\beta\frac{a}{H}\rho\right)-\beta J_0\left(\beta\frac{a}{H}\rho\right)+\gamma_i(\rho,z/H)\,\beta J_0\left(\beta\frac{a}{H}\rho\right)\Big]d\beta \qquad (16c)$$
$$\tau_{zzi}(\rho,z/H)=P_z/\pi a^2\int_0^{\infty}\gamma_i(\rho,z/H)J_0\left(\beta\frac{a}{H}\rho\right)d\beta \qquad (16d)$$
$$\begin{cases} i=1:\ \text{for the layer}\,(z/H<1)\\ i=0:\ \text{for the substrate}\,(z/H>1)\end{cases}$$

Here, Ω_i, Ψ_i, η_i and γ_i are defined in the Appendix. The second invarient of the deviatoric stress tensor, J_2, is defined as

$$J_2=\frac{1}{2}S_{ij}S_{ij} \qquad (17)$$

where,

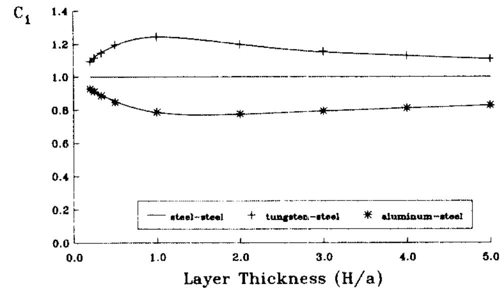


Fig. 2 Relative approach as functions of layer thickness fo various layer-substrate combinations

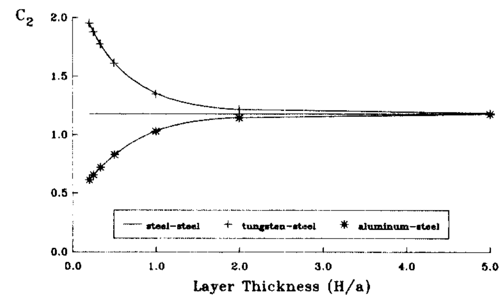


Fig. 3 Radius of contact as functions of layer thickness for various layer-substrate combinations

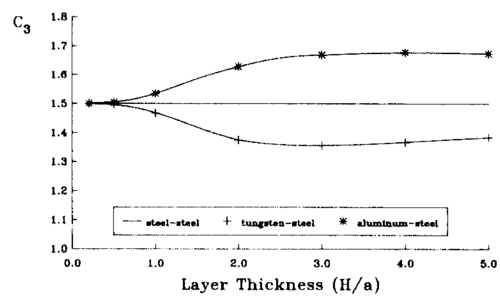


Fig. 4 Peak Surface pressure as function of layer thickness for various layer-substrate combinations

$$S_{ij}=\sigma_{ij}-P_0\delta_{ij} \qquad (18)$$
$$P_0=\frac{1}{3}(\tau_{rri}+\tau_{\theta\theta i}+\tau_{zzi}) \qquad (19)$$

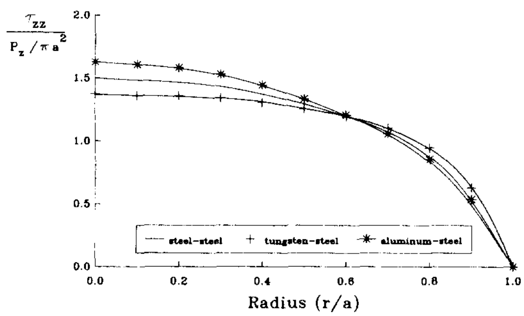
Here, S_{ij} is the deviatoric stress and P_0 is the hydro-static Pressure.

3. NUMERICAL RESULTS

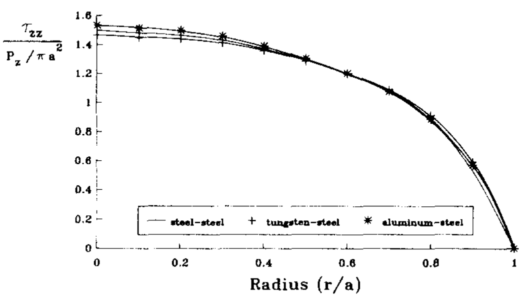
Computations have been carried out for several layer substrate combinations. The thinnest layer had a thickness of one-fifth of the radius of the contact region. The material properties(shear modulus, poisson's ratio) considered in this study are shown in Table 1. The case of tungsten coated to the steel represents the hard coating while the case of the aluminium coated to the steel represents the soft coating. The peak surface stress distributions for various layer-substrate combinations are shown in Fig. 5. The surface stress at the center of the contact for the case of soft coating is more concentrated than the case of hard coating when the layer is thin ($H/a=0.5$). However, both cases produces almost identical surface stress concentrations when the layer gets relatively thick ($H/a=2.0$). The contour for the second invariant of the deviatoric stress tensor normalized with respect to the average contact pressure, $\sqrt{J_2}/P_z/\pi a^2$ are given in Fig. 6, Fig. 7 and Fig. 8. For the case of homogeneous medium (fig. 6) the maximum $\sqrt{J_2}/P_z/\pi a^2$ of 0.528 occurs at $z/a \approx 0.45$. Fig. 7 is for the case of tungsten coated to steel and Fig. 8 is for the case of aluminum coated to steel. The layer thickness

Table 1 Material properties

Materials	Shear Modulus μ(GPa)	Posson's Ratio ν
steel	80	0.28
aluminium	27	0.33
tungsten	160	0.30



(a) H/a=0.5 and



(b) H/a=1.0

Fig. 5 Surface pressure distributions for various layer-substrate combinations

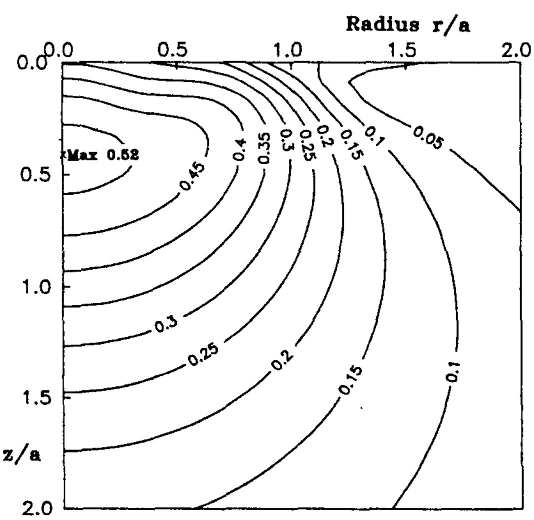
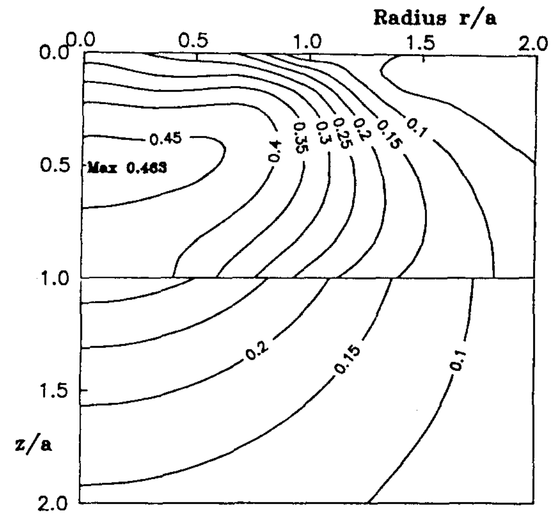
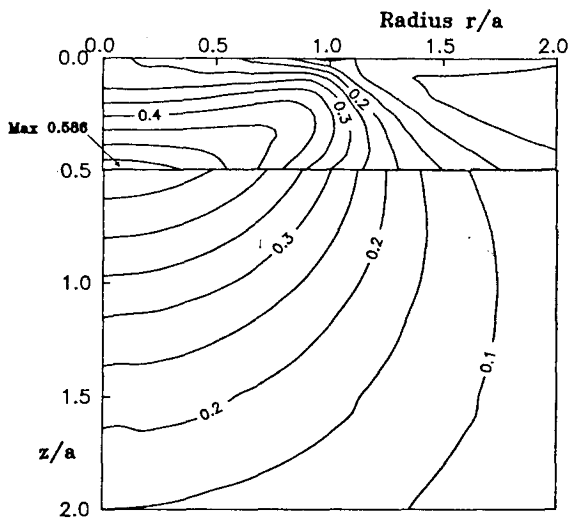


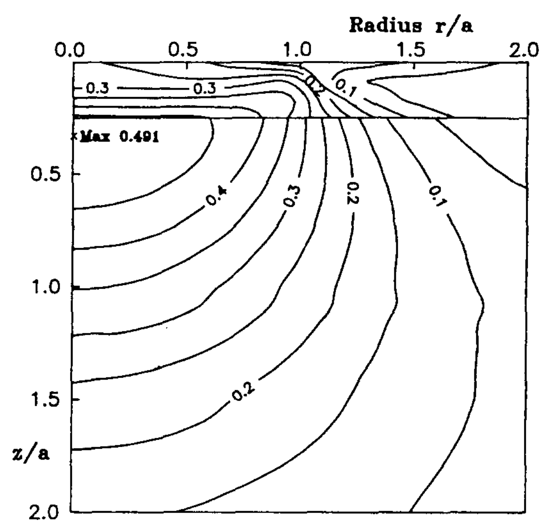
Fig. 6 Contour plots for the second invariant of the deviatoric stress tensor, $\sqrt{J_2}/P_z/\pi a^2$ (layer : steel ; substrate : steel)



(a) $H/a=1.0$

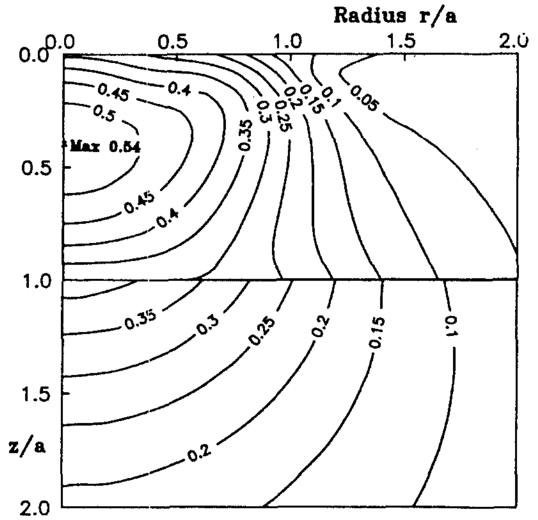


(b) $H/a=0.5$

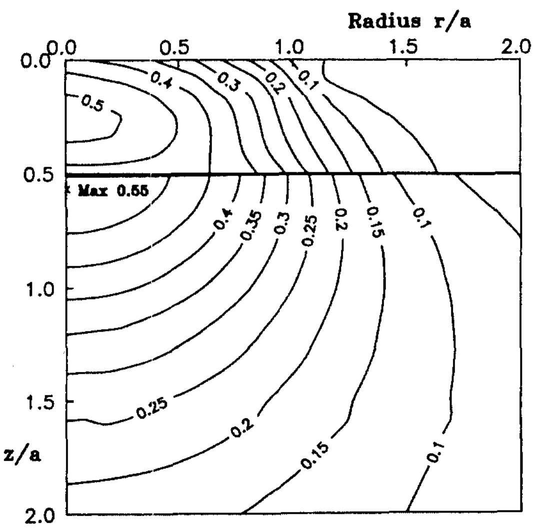


(c) $H/a=0.25$

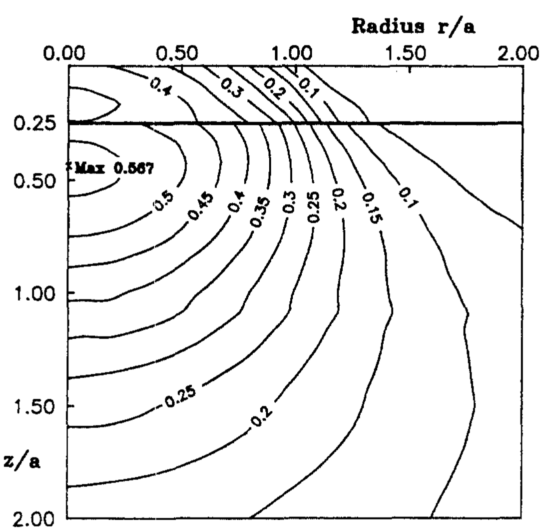
Fig. 7 Contour plots for the second invariant of the deviatoric stress tensor, $\sqrt{J_2}/P_z/\pi a^2$ (layer : tungsten ; substrate : steel)



(a) $H/a=1.0$



(b) $H/a=0.5$



(c) $H/a=0.25$

Fig. 8 Contour plots for the second invariant of the deviatoric stress tensor, $\sqrt{J_2}/P_z/\pi a^2$ (layer : amuminum ; substrate : steel)

H/a is varied from 0.25 to 1.0 for both cases. At the layer-substrate interface, there are considerable jumps for the values of $\sqrt{J_2}/P_z/\pi a^2$ because of material inhomogeneity. The location where the maximum $\sqrt{J_2}/P_z/\pi a^2$ takes place moves from the layer to the substrate as the layer thickness (H/a) decreases.

4. CONCLUSION

A contact problem is formulated and solved numerically when an elastic layer, which is bonded to a sphere, is subjected to a normal loading. The contact stress at the layer surface as well as the Mises stress in the layer and the sphere are calculated for various layer-substrate combinations. The Mises stresses evaluated will give the assessment to pinpoint the most vulnerable area when a coated material is subjected to a compressive loading.

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APPENDIX

Ω_i, Ψ_i, η_i and γ_i in Eq. (16) are defined in the forms,

$$\Omega_1(\beta,z/H)=$$
$$\{A_0(\beta)+\omega D_0(\beta)\}\mathrm{sh}(\omega)+\{B_0(\beta)+\omega C_0(\beta)\}\mathrm{ch}(\omega)$$
$$\Omega_0(\beta,z/H)=$$
$$-\{M_0(\beta)+\omega N_0(\beta)\}e^{-\omega}$$
$$\Psi_1(\beta,z/H)=$$
$$-[\omega D_0(\beta)\mathrm{ch}(\omega)+\{\omega C_0(\beta)-\varepsilon_0(\beta)+D_0(\beta)\}\mathrm{sh}(\omega)]\beta$$
$$\Psi_0(\beta,z/H)=$$
$$-[U_0(\beta)+(1-2\nu_0)V_0(\beta)+\omega V_0(\beta)]\beta e^{-\omega}$$
$$\eta_1(\beta,z/H)=$$
$$-2\nu_1[C_0(\beta)\mathrm{sh}(\beta)+D_0(\beta)\mathrm{ch}(\beta)]\beta$$
$$\eta_0(\beta,z/H)=$$
$$-2\nu_0 N_0(\beta)\beta e^{-\omega}$$
$$\gamma_1(\beta,z/H)=$$
$$-[\omega D_0(\beta)\mathrm{sh}(\omega)+\{\omega C_0(\beta)-\varepsilon_0(\beta)+D_0(\beta)\}\mathrm{ch}(\omega)]\beta$$
$$\gamma_0(\beta,z/H)=$$
$$-[U_0(\beta)+2(1-\nu_0)V_0(\beta)+\omega V_0(\beta)]\beta e^{-\omega}$$

where, $\omega=\beta z/H$.